Frequency-scaling of discrete-time LTI system responses

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Abstract

The article describes a method for approximately preserving the shape of discrete-time LTI system amplitude and phase responses while changing the frequency scale. The typical application cases include cutoff or sampling rate changes. Two versions of the method are presented, a simple and a topology-preserving one.

1 Introduction

The amplitude and phase responses of continuous-time filters, plotted in a logarithmic frequency scale, usually retain their shape if their cutoffs are changed. The discrete-time systems are generally not capable of that, although a reasonably close approximation particularly occurs for systems obtained from continuous-time prototypes by an application of the bilinear transform. Since the responses of the discrete-time systems are a function of the normalized frequency, the same response variation artifacts occur when the system sampling rate is changed.

As we know, quite sometimes in music, the imperfections can become desired features, if they turn out to be nice-sounding. One of the possible sources of such imperfections are the imprecisions of the conversion from continuous to discrete time, the most common case being z^{-1} blocks inserted into delayless signal path. In such cases, these desired imperfections may disappear if the cutoff gets outside of a certain range, or if the system sampling rate is changed.

This article describes a way of preserving such imperfections. Since both the case of a changed cutoff and the case of a changed sampling rate technically imply simply a *rescaling* of the frequency axis, we are going to analyse the problem in the terms of frequency scaling.

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2 Transform method

Remembering that the bilinear transform performs a near-to-perfect mapping of the s-plane frequency axis to the z-plane unit circle we can create an s-plane counterpart of any discrete-time system by applying the inverse bilinear transform[1]:

$$z = \frac{1 + sT/2}{1 - sT/2} \tag{1}$$

The s-plane version of the system can be subjected to the scaling of the frequency axis without any response distortion whatsoever.

$$s = ks' \tag{2}$$

Performing the bilinear transform of the scaled version back into the z-plane:

$$s' = \frac{2}{T} \cdot \frac{z' - 1}{z' + 1}$$

we obtain a frequency-scaled version of the original system. While such frequency scaling is not ideal, due to the obvious frequency axis warping in higher frequencies range, it should be reasonably close.

[1]Notice that such inverse-transfomed continuous-time system is also useful as a general representation of the original discrete-time system, particularly for the purposes of analysis.

The complete transformation is therefore defined by the substitution:

$$z = \frac{z' + \alpha}{\alpha z' + 1} \quad \text{where } \alpha = \frac{1-k}{1+k} \tag{3}$$

Apparently the topology and the state variables of the original system are not preserved by the method, the corresponding information being lost already after the conversion to the s-plane. To ensure that the scaled discrete-time system still has nice modulated-case behavior, one could use some standard topologies for the intermediate continuous-time system, the latter being converted to the discrete-time case using the appropriate methods, particularly the ones described in [1]. The most obvious of such topologies would be a serial biquad decomposition, the biquad sections being implemented in the state-variable form.

3 Prewarping

A common technique to use in conjunction with the bilinear transform is frequency prewarping, to ensure a one-to-one mapping between two particular frequencies. Ordinarily one of these frequencies would be picked in the s-plane, the other in the z-plane, the cutoff being the most common choice. Since (3) performs a mapping from z- to z-plane we are going to have both frequencies in the z-plane. Let the discrete-time normalized frequencies ω_d and ω'_d be two such frequencies for the original and the transformed system respectively. Obviously, the frequency mapping caused by (3) is

$$\tan\frac{\omega_d}{2} = k \tan\frac{\omega'_d}{2} \tag{4}$$

Substituting ω_d and ω'_d into (4) we find the value of k. Thus (4) can be used as a prewarping formula.

As with the ordinary bilinear transform, usually ω_d and ω'_d would be picked up at some characteristic point of the response, the cutoff point being the most obvious option.

4 Preserving the topology

In case the preservation of the topology is not only about nice modulation behavior, but also about keeping other filter characteristics, such as nonlinearities, a modified version of the method may be used.

In [1], the transformation of a continuous-time structure diagram into the discrete-time one was achieved by replacing integrators by their discrete-time models. We can do a similar thing here, by replacing z^{-1} blocks by their continuous-time models, the latter obtained from (1) using e.g. state-variable implementation. The frequency scaling (2) can be achieved by changing the cutoffs of the integrators. The transformation back to the z-plane is performed as described in [1].

It is also possible to perform the just described steps in a somewhat more elegant way. From (1) and (2) we obtain

$$z^{-1} = \frac{1 - ks'T/2}{1 + ks'T/2} \tag{5}$$

Which means that effectively we replace the z^{-1} block by a 1-pole allpass filter with a cutoff $2/(kT)$. Subsequently, (3) performs just a bilinear transform of such allpass filter back to the z-plane .

Therefore, in order to preserve the system topology, we can simply replace all z^{-1} blocks by bilinear-transformed discrete-time models of (5). In order to ensure nice modulated-case behavior such models can be built e.g. by applying the method described in [1] to a state-variable continuous-time implementation of (5).

The allpass filter implementation could also be based directly on (3) using classical discrete-time structure forms, although generally such forms could exhibit inferior behavior in the modulated parameter cases.[2] Notice however, that since all mentioned allpass structures feature a delayless path anyway, application of methods such as described in [1] is still usually required.

[2]The author didn't investigate it in more detail, might as well be that such structures also perform well.

Computing the group delay of (5) we obtain

$$-\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\arg\frac{1-ksT/2}{1+ksT/2}\right)\approx kT \quad \text{for } \omega\ll 1$$

which means that we have just effectively replaced a single-sample delay with a delay by kT. Thus, if the artifacts to be preserved are caused only by a subset of z^{-1} blocks (like e.g. the artifacts caused by insertion of z^{-1} blocks into delayless feedback path), it might be sufficient to replace only those blocks.

5 Application

A number of distinct cases of application of the technique described in this paper are immediately obvious.

Changed sampling rate
There is a sampling rate at which the system has the desired performance, at the same time we don't want to fix the artifacts related to the cutoff changes. A special case of such application is a conversion of an FIR system to other sampling rates.

Changed cutoff
There is a normalized discrete-time cutoff frequency, at which the system has exactly the desired response shapes, other cutoffs featuring unwanted artifacts. Apparently, we also want to counteract the sampling rate-related artifacts. Notice that this case is not 100% restricted to cutoff changes, variations of other kinds of system parameters can be handled as well.

Mixed case
There is a range of normalized discrete-time cutoffs, the response shapes within which we wish to extend to the full range of possible cutoff values.

6 Conclusion

The described methods allows to approximately preserve the shapes of the amplitude and phase response of the discrete-time LTI systems in the cases of frequency axis scaling, such as cutoff and sampling rate changes. A version of the technique preserving the system topology has been additionally presented. The application of method includes cases when the preservation of imperfections of certain discrete-time systems are desired.

Fig. 1 contains an example of a result of the application of the described method.

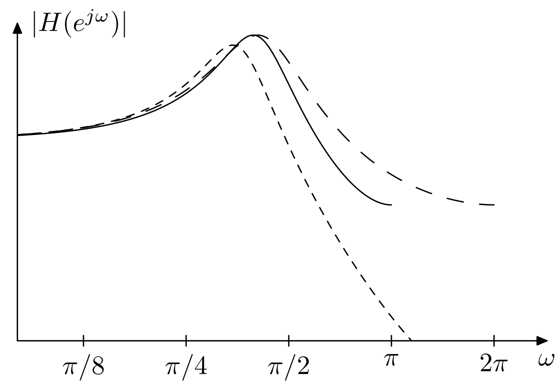

Figure 1: Application of the method to a discrete-time LTI system. The solid line shows the original response. The fine-dashed line shows the result of doubling the sampling rate without applying the method. The large-dashed line shows the result of doubling the sampling rate with applying the method. The frequency scale is normalized against the original sampling rate.

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References

[1] Zavalishin V. "Preserving the LTI system topology in s- to z-plane transforms"