Generation of bandlimited sync transitions for sine waveforms

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Abstract

A common way to generate bandlimited sync transitions for the classical analog waveforms is the BLEP method. The method works well for the sawtooth, pulse, and triangle waveforms, however fails in case of the sine. A number of methods to overcome that problem is proposed in the article.

1 Introduction

The BLEP method (including its minimum-phase variant) [1] provides a way for generating bandlimited zero-order discontinuities (discontinuities of the signal), which can be applied for generation of bandlimited saw and pulse waveforms and also sync transitions in such waveforms. A triangle waveform contains first-order discontinuities (discontinuities of the first derivative) plus zero-order discontinuities which may appear during syncing, therefore in addition to the "zero-order" BLEPs one also needs to use the "first-order" BLEPs.

A sine waveform on its own doesn't require any bandlimiting (provided the sine frequency lies below the Nyquist limit), however a sync transition generates discontinuities of all orders up to infinity, therefore an infinite number of BLEPs of all possible orders is theoretically required. In the present article we will discuss a number of ways to deal with that problem.

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2 Problem specification

We are going to consider only the constant-frequency sine waveforms, in which case the sync transition is simply a phase discontinuity. Let

$$x_1(t) = \cos(\omega t + \varphi_1) = \frac{X_1(t) + X_1^*(t)}{2}$$
$$x_2(t) = \cos(\omega t + \varphi_2) = \frac{X_2(t) + X_2^*(t)}{2}$$

be the signals before and after the transition and let

$$X_1(t) = e^{j(\omega t + \varphi_1)}$$
$$X_2(t) = e^{j(\omega t + \varphi_2)}$$

be their analytic counterparts (the asterisk in $X^*(t)$ denotes the complex conjugation). Let

$$h(t) = \frac{1}{2}\operatorname{sgn} t = \begin{cases} -\frac{1}{2} & \text{if } t < 0 \\ 0 & \text{if } t = 0 \\ \frac{1}{2} & \text{if } t > 0 \end{cases}$$

be the step function and let the sync transition occur at $t = 0$. Then the output signal can be represented as

$$\begin{aligned} y(t) &= \left(\frac{1}{2} - h(t)\right) x_1(t) + \left(\frac{1}{2} + h(t)\right) x_2(t) \\ &= \frac{x_1(t) + x_2(t)}{2} + h(t)\big(x_2(t) - x_1(t)\big) \\ &= x(t) + h(t)\Delta x(t) \end{aligned} \quad (1)$$

where

$$x(t) = \frac{x_1(t) + x_2(t)}{2}$$

is the average of the two signals and

$$\Delta x(t) = x_2(t) - x_1(t)$$

is their difference. Respectively,

$$\Delta X(t) = X_2(t) - X_1(t)$$

is the difference of the analytic signals.

We want to bandlimit the output signal $y(t)$ to the frequency band from -1 to 1. Apparently, we can assume the $(-1, 1)$ band without loss of generality, as this means simply a choice of the frequency units. The upper bound of the frequency band would typically correspond to the Nyquist frequency, however doesn't have to, e.g. one could specify a bound slightly below or slightly above the Nyquist limit (whatever "slightly" means). We will also assume that the signals x_1 and x_2 are already bandlimited ($|\omega| < 1$).

The bandlimited versions of the originally non-bandlimited signals will be denoted by a bar on top, e.g. $\bar{y}(t)$. Note that some signals will be bandlimited to other bands than $(-1, 1)$, in which case we will explicitly mention so.

3 Notation issues

In the text of this article we will need to deal with the integral exponents, sines, and cosines. A number of different notations are apparently in use here [2]. In order to simplify the formulas in the article we will use the following somewhat non-standard notation.

Restricting z to taking only purely imaginary values $z = j\varphi$ we let

$$\begin{aligned}
\operatorname{Ein} z &= \int_0^z \frac{e^t - 1}{t}\,\mathrm{d}t \\
\operatorname{Ei} z &= \gamma + \ln|z| + \operatorname{Ein} z \\
E_n(z) &= \int_1^\infty \frac{e^{zt}}{t^n}\,\mathrm{d}t \qquad (n \geq 1) \\
\operatorname{ei} j\varphi &= \operatorname{Ei} j\varphi - j\frac{\pi}{2}\operatorname{sgn}\varphi = -E_1(j\varphi)
\end{aligned}$$

where the absolute value of z in $\ln|z|$ and the term $(j\pi/2)\operatorname{sgn}\varphi$ ensure the conjugate symmetries $\operatorname{Ei}(-j\varphi) = \operatorname{Ei}^*(j\varphi)$ and $\operatorname{ei}(-j\varphi) = \operatorname{ei}^*(j\varphi)$ respectively, and γ denotes the Euler's constant. Thus:

$$\begin{aligned}
\operatorname{Cin}\varphi &= \int_0^\varphi \frac{\cos t - 1}{t}\,\mathrm{d}t \\
\operatorname{Ci}\varphi &= \gamma + \ln|\varphi| + \operatorname{Cin}\varphi \\
\operatorname{Si}\varphi &= \int_0^\varphi \frac{\sin t}{t}\,\mathrm{d}t \\
\operatorname{si}\varphi &= \operatorname{Si}\varphi - \frac{\pi}{2}\operatorname{sgn}\varphi
\end{aligned}$$

Therefore

$$\begin{aligned}
\operatorname{Ein} j\varphi &= \operatorname{Cin}\varphi + j\operatorname{Si}\varphi \\
\operatorname{Ei} j\varphi &= \operatorname{Ci}\varphi + j\operatorname{Si}\varphi \\
\operatorname{ei} j\varphi &= \operatorname{Ci}\varphi + j\operatorname{si}\varphi \\
\operatorname{Ci}(-\varphi) &= \operatorname{Ci}\varphi \\
\operatorname{Si}(-\varphi) &= -\operatorname{Si}\varphi \\
\operatorname{si}(-\varphi) &= -\operatorname{si}\varphi \\
\operatorname{Ein}(-j\varphi) &= \operatorname{Ein}^* j\varphi \\
\operatorname{Ei}(-j\varphi) &= \operatorname{Ei}^* j\varphi \\
\operatorname{ei}(-j\varphi) &= \operatorname{ei}^* j\varphi \\
E_n(-j\varphi) &= E_n^*(j\varphi) \\
nE_{n+1}(z) &= zE_n(z) + e^z \\
\frac{\mathrm{d}}{\mathrm{d}z}E_n(z) &= E_{n-1}(z)
\end{aligned}$$

The limit values are

$$\begin{aligned}
\operatorname{Ci}(\pm\infty) &= 0 \\
\operatorname{Si}(\pm\infty) &= \pm\frac{\pi}{2} \\
\operatorname{ei}(\pm\infty \cdot j) &= 0 \\
E_n(\pm\infty \cdot j) &= 0 \\
E_{n+1}(0) &= \frac{1}{n}
\end{aligned}$$

A robust numerical method to compute $\operatorname{Ci}\varphi$ and $\operatorname{Si} t$ can be found in [3]. Essentially, it is using the series expansion if $|\varphi| < 2$ and the continued fraction for $E_1(z)$ otherwise, both of which can be also found in [2]. The remaining functions can be built from $\operatorname{Ci}\varphi$ and $\operatorname{Si}\varphi$. Note that in order to compute $\operatorname{Cin}\varphi$ and $\operatorname{Ein} j\varphi$ it is reasonable to use the series expansion for $\operatorname{Cin}\varphi$ instead of the one for $\operatorname{Ci}\varphi$. Also, keep in mind the notation specifics of the present article!

4 Multiple BLEPs method

If the sine frequency lies well below the Nyquist limit, then we would intuitively expect higher-order BLEPs to have rather little effect on the waveform, in which case we may simply discard them. Let's analyse this question in more detail.

4.1 Fundamentals

Recalling that

$$h(t) = \frac{1}{2}\operatorname{sgn} t = \int_0^t \delta(\tau)\,\mathrm{d}\tau$$

we can refer to $h(t)$ as to the zero-order step function, and to $\delta(t)$ as to the -1st-order step function, using notations $h_0(t) = h(t)$ and $h_{-1}(t) = \delta(t)$, whereas the higher-order step functions are defined as

$$h_n(t) = \int_0^t h_{n-1}(\tau)\,\mathrm{d}\tau = \frac{t^n}{2n!}\operatorname{sgn} t$$

We define the aliasing residual of n-th order as

$$\Delta\bar{h}_n(t) = h_n(t) - \bar{h}_n(t)$$

Then the bandlimited output signal is

$$\bar{y}(t) = y(t) - \Delta\bar{y}(t) = y(t) - \sum_{n=0}^{\infty} \Delta y^{(n)} \Delta\bar{h}_n(t)$$

where

$$\Delta y^{(n)} = y^{(n)}(t+0) - y^{(n)}(t-0)$$

4.2 Analytic BLEPs

Writing the Fourier integral for $h_0(t)$ we have

$$\begin{aligned} h_0(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{j\omega t}}{j\omega}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\lim_{\varepsilon\to+0}\left(\int_{-\infty}^{-\varepsilon} \frac{e^{j\omega t}}{j\omega}\,\mathrm{d}\omega + \int_{\varepsilon}^{\infty} \frac{e^{j\omega t}}{j\omega}\,\mathrm{d}\omega\right) \\ &= \frac{1}{2\pi j}\lim_{\varepsilon\to+0}\big(E_1(j\varepsilon t) - E_1(-j\varepsilon t)\big) \end{aligned}$$

Thus

$$h_0(t) = \lim_{\varepsilon\to+0} \frac{H_0(t) + H_0^*(t)}{2}$$

where

$$H_0(t) = \frac{1}{\pi j}E_1(j\varepsilon t) = \frac{1}{\pi j}\int_{\varepsilon}^{\infty} \frac{e^{j\omega t}}{j\omega}\,\mathrm{d}\omega$$

Therefore, $H_0(t)$ can be considered as an analytic counterpart of $h_0(t)$ (except that it's parametrized with ε). Bandlimiting the signal $H_0(t)$:

$$\bar{H}_0(t) = \frac{1}{\pi j}\int_{\varepsilon}^{1} \frac{e^{j\omega t}}{j\omega}\,\mathrm{d}\omega = \frac{1}{\pi j}E_1(j\omega t)\Big|_{\omega=1}^{\varepsilon}$$

we obtain the expression for the aliasing residual

$$\begin{aligned} \Delta\bar{H}_0(t) = H_0(t) - \bar{H}_0(t) &= \frac{1}{\pi j}\int_{1}^{\infty} \frac{e^{j\omega t}}{j\omega}\,\mathrm{d}\omega \\ &= \frac{1}{\pi j}E_1(jt) = H_0(t)\Big|_{\varepsilon=1} \end{aligned}$$

Thus, $\Delta\bar{H}_0(t)$ doesn't depend on ε and

$$\Delta\bar{h}_0(t) = \frac{\Delta\bar{H}_0(t) + \Delta\bar{H}_0^*(t)}{2}$$

Now consider

$$H_1(t) = \bar{H}_1(t) + \Delta\bar{H}_1(t)$$

which can be alternatively written as

$$\int_0^t H_0(\tau)\,\mathrm{d}\tau = \int_0^t \bar{H}_0(\tau)\,\mathrm{d}\tau + \int_0^t \Delta\bar{H}_0(\tau)\,\mathrm{d}\tau$$

Evaluating the last term we get

$$\begin{aligned} \Delta\bar{H}_1(t) = \int_0^t \Delta\bar{H}_0(\tau)\,\mathrm{d}\tau &= \frac{1}{\pi j}\int_0^t E_1(j\tau)\,\mathrm{d}\tau \\ &= -\frac{E_2(j\tau)}{\pi}\Big|_{\tau=0}^{t} = -\frac{E_2(jt)}{\pi} + \frac{1}{\pi} \end{aligned}$$

The above is not the most desirable scenario, as we would rather have $\Delta\bar{H}_1(t) \to 0$ for $t \to \infty$. Notice however, that the term $1/\pi$ is simply a DC offset, therefore it is bandlimited and can be simply included into $\bar{H}_1$:

$$\bar{H}_1(t) = \int_0^t \bar{H}_0(\tau)\,\mathrm{d}\tau + \frac{1}{\pi}$$

$$\Delta\bar{H}_1(t) = -\frac{E_2(jt)}{\pi}$$

Continuing in the same manner, we obtain

$$\Delta\bar{H}_n(t) = \frac{E_{n+1}(jt)}{\pi j^{n+1}}$$

which is a general expression for an n-th order bandlimited analytic aliasing residue.

4.3 Real-domain BLEPs

The real domain expression for the aliasing residuals can be obtained as

$$\begin{aligned}\Delta\bar{h}_n(t) &= \frac{\Delta\bar{H}_n(t) + \Delta\bar{H}^*_n(t)}{2} \\ &= \frac{E_{n+1}(jt) + (-1)^{n+1}E^*_{n+1}(jt)}{2\pi j^{n+1}}\end{aligned} \tag{2}$$

Notably, for $n = 0$ the above turns into

$$\Delta\bar{h}_0(t) = \frac{E_1(jt) - E^*_1(jt)}{2\pi j} = \frac{1}{2}\operatorname{sgn} t - \frac{1}{\pi}\operatorname{Si} t \tag{3}$$

which is the familiar expression for the zero-order aliasing residual.

The problem with (2) however, is that the function $E_1(jt)$, which is required to recursively evaluate $E_n(jt)$, is getting infinitely large around $t = 0$. Therefore, for small t one should use (3) at $n = 0$ and take special care in the evaluation of the term $jtE_1(jt)$ in the expression for $E_2(jt)$ at $n > 0$.

A better option is to notice that

$$\begin{aligned}n\Delta\bar{h}_n(t) &= n\frac{E_{n+1}(jt) + (-1)^{n+1}E^*_{n+1}(jt)}{2\pi j^{n+1}} \\ &= jt\frac{E_n(jt) + (-1)^{n+1}(-jt)E^*_{n+1}(jt)}{2\pi j^{n+1}} \\ &\qquad + \frac{e^{jt} + (-1)^{n+1}e^{-jt}}{2\pi j^{n+1}} \\ &= t\Delta\bar{h}_{n-1}(t) + \frac{1}{\pi}\sin^{(-n)} t\end{aligned}$$

where $\sin^{(-n)} t$ is the n-th antiderivative of $\sin t$:

$$\sin^{(-n)} t = \begin{cases}(-1)^{(n+1)/2}\cos t & \text{for odd } n \\ (-1)^{n/2}\sin t & \text{for even } n\end{cases}$$

which allows us to compute $\Delta\bar{h}_n(t)$ from $\Delta\bar{h}_0(t)$.

Note that it is also not difficult to obtain the explicit form of $\bar{h}(t)$:

$$\bar{h}_n(t) = \frac{1}{\pi}\operatorname{Si}^{(-n)}(t)$$

where $\operatorname{Si}^{(-n)} t$ is the n-th antiderivative of $\operatorname{Si} t$, which is computed as

$$n\operatorname{Si}^{(-n)} t = t\operatorname{Si}^{(-n+1)} t - \sin^{(-n)} t$$

4.4 Error estimation

Now we will estimate the bounds for $\Delta\bar{h}_n(t)$. First, we notice that for $n = 0$ we have

$$\left|\Delta\bar{h}_0(t)\right| = \left|h_0(t) - \frac{1}{\pi}\operatorname{Si} t\right| \le \frac{1}{2} = \left|\Delta\bar{h}_0(0)\right|$$

For higher orders, using (2) we have

$$\begin{aligned}\left|\Delta\bar{h}_n(t)\right| &= \left|\frac{E_{n+1}(jt) + (-1)^{n+1}E^*_{n+1}(jt)}{2\pi j^{n+1}}\right| \\ &= \frac{1}{2\pi}\left|\int_1^\infty \frac{e^{j\omega t} + (-1)^{n+1}e^{-j\omega t}}{\omega^{n+1}}\,\mathrm{d}\omega\right| \\ &\le \frac{1}{\pi}\int_1^\infty \frac{\mathrm{d}\omega}{\omega^{n+1}} = \frac{1}{\pi n} = \left|\Delta\bar{h}_n(0)\right|\end{aligned}$$

Now assume, that instead of the infinite series we use an approximation for $\Delta\bar{y}$:

$$\Delta\bar{y} \approx \sum_{n=0}^{N} \Delta y^{(n)}\Delta\bar{h}_n(t)$$

The error of the approximation is given by the remainder of the series:

$$\varepsilon = \left|\sum_{n=N+1}^{\infty} \Delta y^{(n)}\Delta\bar{h}_n(t)\right| \le \sum_{n=N+1}^{\infty} \left|\Delta y^{(n)}\Delta\bar{h}_n(t)\right|$$

Noticing that $\left|\Delta y^{(n)}\right| \le 2\left|\omega\right|^n$, and recalling the bound estimations for $\Delta\bar{h}_n(t)$ we have

$$\varepsilon \le \sum_{n=N+1}^{\infty} 2\left|\omega\right|^n\left|\Delta\bar{h}_n(0)\right| = \sum_{n=N+1}^{\infty} \frac{2\left|\omega\right|^n}{\pi n}$$

Therefore if $|\omega| \ll 1$ then $\varepsilon \approx 0$.

4.5 Windowing

In choosing N one should take into account the fact that the BLEP method doesn't provide the full bandlimiting anyway, as each of the aliasing residuals is infinitely long, and some windowing would be used in practice. Therefore the number of BLEPs and the window size have to be chosen together, as to provide comparable levels of "imperfectness".

The windowed BLEPs can be constructed in a number of ways. The most obvious way is to window the aliasing residuals directly:

$$\left(\Delta\bar{h}_n\right)_{\mathrm{TL}}(t) = W(t)\Delta\bar{h}_n(t)$$

where $W(t)$ is the window function, and $(\,)_{\mathrm{TL}}$ denotes the timelimited version of a function. Note that an important side effect of windowing is that the argument of $\Delta\bar{h}(t)$ becomes restricted to the window size, and thus the functions $\left(\Delta\bar{h}_n\right)_{\mathrm{TL}}(t)$ can be simply tabulated.

Another approach can be to window the -1th-order BLEP:

$$\left(\bar{h}_{-1}\right)_{\mathrm{TL}}(t) = W(t)\bar{h}_{-1}(t) = W(t)\frac{\operatorname{sinc} t}{\pi}$$

and compute the rest by the recursive integration of the aliasing residuals:

$$\left(\Delta\bar{h}_n\right)_{\mathrm{TL}}(t) = a_n\int_0^t \left(\Delta\bar{h}_{n-1}\right)_{\mathrm{TL}}(\tau)\,\mathrm{d}\tau + c_n$$

where the constants a_n and c_n should be chosen so as to satisfy the requirement $\left(\Delta\bar{h}_n\right)_{\mathrm{TL}}(\pm\infty) = 0$.

The latter approach might not seem very reasonable, because of a large amount of numerical integration involved, which should lead to increasing precision losses. However, there is a special case (particularly useful at small window sizes) where the integration can be done analytically. Recall that the (continuous-time) impulse responses of symmetric interpolators are typically approximations of the sinc function [4] [5], and thus one may take such impulse response as $\left(\bar{h}_{-1}\right)_{\mathrm{W}}(t)$. Then one can construct higher-order BLEPs from $\left(\bar{h}_{-1}\right)_{\mathrm{W}}(t)$ as described above.

5 Ring modulation method

Consider again (1). Obviously, $x(t)$ is already bandlimited, therefore we only need to bandlimit the remainder $h(t)\Delta x(t)$ (remember that $h(t) = h_0(t)$).

Imagine that instead of $h(t)$ we have used in (1) its bandlimited (to $(-1, 1)$) version $\bar{h}(t)$. Then the remainder of (1) turns into

$$\bar{h}(t)\Delta x(t) = \frac{1}{2}\bar{h}(t)\Delta X(t) + \frac{1}{2}\bar{h}(t)\Delta X^*(t) \tag{4}$$

Recalling that

$$\Delta X(t) = e^{j\omega t}\left(e^{j\varphi_2} - e^{j\varphi_1}\right)$$

and considering the first term of (4) alone, we notice that the multiplication by $X(t)$ shifts the spectrum of $\bar{h}(t)$ to the right by ω. Similarly, in the second term the spectrum of $\bar{h}(t)$ is shifted to the left by ω.

Therefore, if $|\omega| \ll 1$, then the spectrum of $\bar{h}\Delta x$ will have only minor differences to the ideally bandlimited $h\Delta x$. Already if $|\omega| < 1/2$, then all "wrong" partials (that including the aliasing and the missing partials) will be located above $1/2$. Therefore, if e.g. the sampling rate is 88kHz and the sine frequencies are limited to 22kHz, then all the "wrong" partials will be located above 22kHz. Considering that the sampling rate of 88kHz or higher may be desirable for a number of other reasons anyway, the above method might be an acceptable option.

The time-limited version of $\bar{h}(t)$ can be built from the time-limited residual:

$$\left(\bar{h}\right)_{\mathrm{TL}}(t) = h(t) - \left(\Delta\bar{h}\right)_{\mathrm{TL}}(t)$$

or by direct integration of $\left(\bar{h}_{-1}\right)_{\mathrm{TL}}(t)$:

$$\left(\bar{h}\right)_{\mathrm{TL}}(t) = \frac{\int_0^t \left(\bar{h}_{-1}\right)_{\mathrm{TL}}(\tau), \mathrm{d}\tau}{\int_{-\infty}^{\infty} \left(\bar{h}_{-1}\right)_{\mathrm{TL}}(\tau), \mathrm{d}\tau}$$

It is convenient to write the band- and time-limited version of (1) as

$$\begin{aligned}\bar{y}(t) &= x(t) + \left(\bar{h}\right)_{\mathrm{TL}}(t)\Delta x(t)\\ &= y(t) - \left(\Delta\bar{h}\right)_{\mathrm{TL}}(t)\Delta x(t)\end{aligned}$$

because the latter form, by using the residual, allows intuitive handling of time-overlapping transitions.

6 Frequency shifting method

Now we are going to try to bandlimit $h\Delta x$ to exactly $(-1, 1)$. We'll do so by bandlimiting each of the components $h\Delta X$ and $h\Delta X^*$ to $(-1, 1)$. In order to do that we need to bandlimit $h(t)$ in $h\Delta X$ to $(-\omega - 1, -\omega + 1)$, and bandlimit $h(t)$ in $h\Delta X^*$ to $(\omega - 1, \omega + 1)$.

6.1 Asymmetric BLEP

First, we are going to build an analytic expression for the step function $h(t)$ bandlimited to the frequency range (ω_-, ω_+), where we will assume $\omega_- < 0 < \omega_+$.

Writing the Fourier integral we get

$$\begin{aligned}\bar{h}(t) &= \frac{1}{2\pi}\int_{\omega_-}^{\omega_+} \frac{e^{j\omega t}}{j\omega}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{\omega_-}^{\omega_+} \frac{1}{j\omega}\,\mathrm{d}\omega + \frac{1}{2\pi}\int_{\omega_-}^{\omega_+} \frac{e^{j\omega t}-1}{j\omega}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi j}\left(\ln|\omega| + \operatorname{Ein} j\omega t\right)\Big|_{\omega=\omega_-}^{\omega_+} \\ &= \frac{1}{2\pi j}\operatorname{Ei} j\omega t\Big|_{\omega=\omega_-}^{\omega_+} \end{aligned} \tag{5}$$

where the latter transformation is done noticing that

$$\ln|\omega_+| - \ln|\omega_-| = \ln|j\omega_+ t| - \ln|j\omega_- t|$$

6.2 Putting the things together

Using (5), the bandlimited signal $h\Delta x$ is built as the average of signals $\bar{h}\Delta X$ and $\bar{h}\Delta X^*$, where in the first case $\bar{h}$ is limited to a different frequency band than in the second case:

$$\begin{aligned}\overline{h\Delta x} &= \frac{1}{2}\bar{h}\Delta X + \frac{1}{2}\bar{h}\Delta X^* \\ &= \frac{\Delta X}{4\pi j}\big(\operatorname{Ei} j(-\omega+1)t - \operatorname{Ei} j(-\omega-1)t\big) \\ &\quad + \frac{\Delta X^*}{4\pi j}\big(\operatorname{Ei} j(\omega+1)t - \operatorname{Ei} j(\omega-1)t\big)\end{aligned}$$

Let

$$\begin{aligned}\omega_L &= 1 - \omega \\ \omega_H &= 1 + \omega\end{aligned} \tag{6}$$

Note that since $|\omega| < 1$, it means that

$$\begin{aligned}|\omega_L| &\le 2 \\ |\omega_H| &\le 2\end{aligned}$$

(this will be important for estimating the table size for the functions involved in the resulting formula). Using the notation (6) we can write

$$\begin{aligned}\overline{h\Delta x} &= \frac{\Delta X}{4\pi j}\big(\operatorname{Ei} j\omega_L t - \operatorname{Ei}(-j\omega_H t)\big) \\ &\quad + \frac{\Delta X^*}{4\pi j}\big(\operatorname{Ei} j\omega_H t - \operatorname{Ei}(-j\omega_L t)\big) \\ &= \frac{\Delta X}{4\pi j}\big(\operatorname{Ei} j\omega_L t - \operatorname{Ei}(-j\omega_H t)\big) \\ &\quad - \frac{\Delta X^*}{4\pi j}\big(\operatorname{Ei}^* j\omega_L t - \operatorname{Ei}^*(-j\omega_H t)\big) \\ &= \frac{1}{2\pi}\operatorname{Im}\Big(\Delta X(t)\big(\operatorname{Ei} j\omega_L t - \operatorname{Ei}(-j\omega_H t)\big)\Big) \\ &= \frac{1}{2\pi}\operatorname{Im}\big(\Delta X(t)\operatorname{Ei} j\omega t\big)\Big|_{\omega=-\omega_H}^{\omega_L} \\ &= \frac{1}{2\pi}\operatorname{Im}\left(\Delta X(t)\left(\ln\left|\frac{\omega_L}{\omega_H}\right|\right.\right. \\ &\qquad\qquad \left.\left. + \operatorname{Ein} j\omega t\Big|_{\omega=-\omega_H}^{\omega_L}\right)\right)\end{aligned}$$

which gives us a formula for bandlimiting $y(t)$. Note that if $|\omega| \ll 1$, then $\omega_L \approx \omega_H \approx 1$ and

$$\begin{aligned}\overline{h\Delta x} &\approx \frac{1}{2\pi}\operatorname{Im}\big(\Delta X \cdot 2j\operatorname{Si} t\big) \\ &= \frac{\operatorname{Si} t}{\pi}\operatorname{Re}\Delta X = \frac{\operatorname{Si} t}{\pi}\Delta x\end{aligned}$$

which is the ring modulation method's formula.

6.3 Windowing

Let

$$\bar{h}(t) = \frac{1}{2\pi j}\left(\ln\left|\frac{\omega_L}{\omega_H}\right| + \operatorname{Ein} j\omega t\Big|_{\omega=-\omega_H}^{\omega_L}\right)$$

so that

$$\overline{h\Delta x}(t) = \operatorname{Re}\left(\bar{h}(t)\Delta X(t)\right)$$

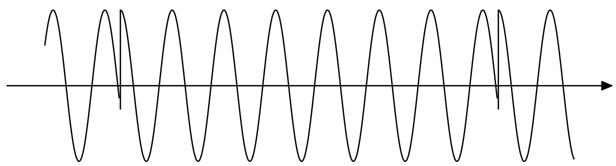

Figure 1: Non-bandlimited signal.

Then

$$\begin{aligned}\bar{y}(t) &= x(t) + \overline{h\Delta x} \\ &= x(t) + h(t)\Delta x(t) - h(t)\Delta x(t) + \overline{h\Delta x}(t) \\ &= y(t) - \mathrm{Re}\big(h(t)\Delta X(t)\big) + \mathrm{Re}\big(\bar{h}(t)\Delta X(t)\big) \\ &= y(t) - \mathrm{Re}\big((h(t) - \bar{h}(t))\Delta X(t)\big) \\ &= y(t) - \mathrm{Re}\big(\Delta\bar{h}(t)\Delta X(t)\big)\end{aligned}$$

Noticing that $\Delta\bar{h}(\pm\infty) = 0$, we conclude that we can timelimit the transition by simply timelimiting the signal $\Delta\bar{h}(t)$.

Note that in the frequency shifting method the function $\bar{h}(t)$ depends on the signal's frequency ω, therefore one can only tabulate $\mathrm{Ein}\, j\varphi$ and the window function, whereas the application of the window must be done in real time.

7 Comparing the spectra

In order to compare the methods a test case was created, where the master oscillator frequency was set to $\omega_M = 0.05$, and the slave oscillator frequency was set to $\omega_S = 7.3\omega_M$. The reset phase was $\varphi = 0$ (Fig. 1). A 20 samples-long Kaiser window with $\alpha = 4$ was used for the transition time limiting. The resulting spectra (generated numerically for a continuous-time case) are plotted in Fig. 2 through Fig. 2d. The dashed lines correspond to the non-bandlimited spectrum.

8 Conclusion

We have described three different methods for generating a bandlimited sync transition in the sine waveform. The third method (frequency shifting) potentially can generate an almost ideally bandlimited

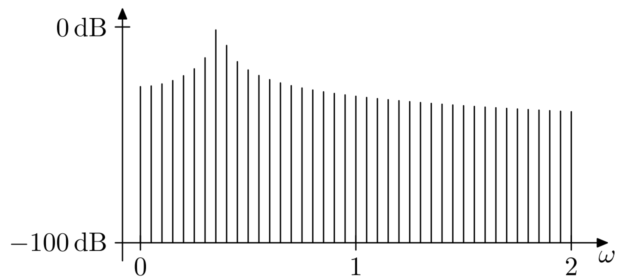

Figure 2: Non-bandlimited signal's spectrum.

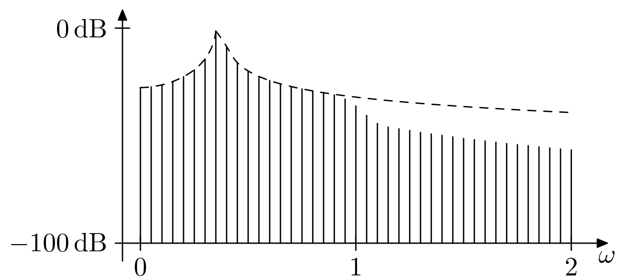

Figure 2a: Single-BLEP method.

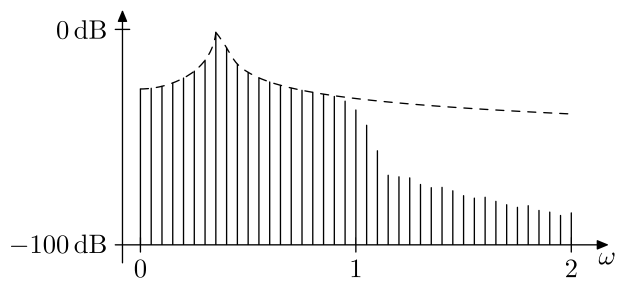

Figure 2b: Multiple BLEPs ($N = 2$) method.

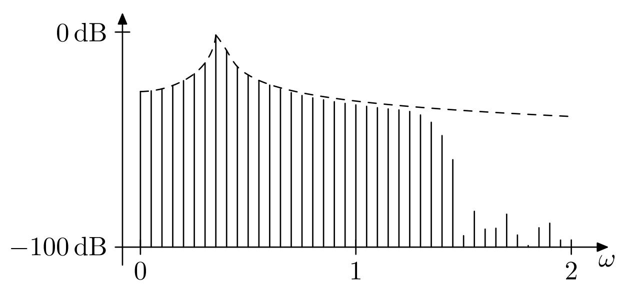

Figure 2c: Ring modulation method.

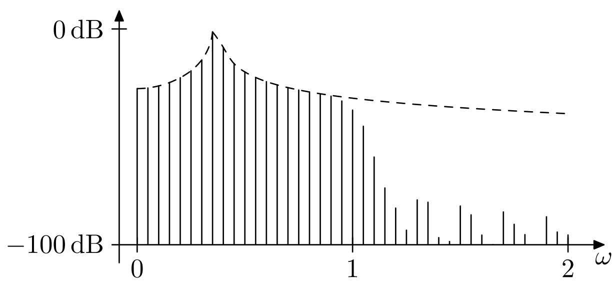

Figure 2d: Frequency shifting method.

transition, given a long enough window. However it requires the computation of about 6 additional tabulated functions (2 times the real and imaginary parts of the integral exponent, the window function, and the imaginary counterpart of the sine waveform). At a similar computation complexity we could have used the multiple BLEPs method to the 5th order (inclusive) instead.

Another limitation of the frequency shifting method is that it is not immediately offering a conversion option to the minimum-phase version. The linear phase bandlimiting requires a lookahead, which means that a latency needs to be introduced in the output. However, often the latency introduced by the necessary lookahead can be either tolerated or compensated by introducing an equal latency in the parallel signal paths. (Also, personally, the author prefers the linear phase antialiasing, because it preserves the phase relationships in the waveform, particularly, the generated waveforms are practically identical to the sampled analog waveforms).

Note that all three methods tend to produce larger errors as the signal's frequency gets closer to the band limit.

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(The http links are at the time of writing the article)